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CITATION:

Arakawa, Tsuneo. Selberg trace formula and Jacobi forms. 数理解析研究所講究録 1991, 752: 1-10

ISSUE DATE:

1991-05

URL:

http://hdl.handle.net/2433/82076

数理解析研究所講究録
第 752 巻 1991 年 1-10

Selberg trace formula and Jacobi forms

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1 Introduction

In this note we present a calculation of the traces of Hecke operators acting on the spaces of Jacobi forms via the general Selberg trace formula. We can represent those traces in a closed form with the use of some arithmetic quantities and the residues at poles of certain Selberg type zeta functions. The calculation of those traces has been done by Skoruppa-Zagier ([S-Z1, 2]) in some cases in a different manner. They have employed the Bergman kernel functions for the spaces of Jacobi forms and also some results of Shimura [Sh] concerning modular forms of half integral weight. Here we use the general Selberg trace formula due originally to Selberg [Se] and to Hejhal [He], Fischer [Fi]. For our calculation we exclusively follow Fischer [Fi].
In this short survey we exhibit only the results which is a generalization of our previous work [Ar] and we shall give a proof in another occasion in details.

2 Jacobi forms and Hecke operators

We use the symbol $e(\alpha)$ as an abbreviation of $\exp(2\pi i\alpha)$. Let l be a positive integer. Let $G^J_{\mathbf{Q}}$ be the Jacobi group defined over $\mathbf{Q}$:

$$G^J_{\mathbf{Q}} = \{(g, (\lambda, \mu), \rho) \mid g \in \mathbf{Q}^l, \lambda, \mu \in \mathbf{Q}^l, \rho \in Sym_l(\mathbf{Q})\},$$

where $\mathbf{Q}^l$ (resp. $Sym_l(\mathbf{Q})$) denotes the space of rational column vectors (resp. rational symmetric matrices) of size l. The composition law of $G^J_{\mathbf{Q}}$ is given by

$$g_1g_2 = (M_1M_2, (\lambda_1, \mu_1)M_2 + (\lambda_2, \mu_2), \rho_1 + \rho_2 - \mu_1{}^t\lambda_1 + \mu^{*t}\lambda^* + \lambda^{*t}\mu_2 + \mu_2{}^t\lambda^*)$$

$$(g_j = (M_j, (\lambda_j, \mu_j), \rho_j) \in G^J_{\mathbf{Q}},\ j = 1, 2)$$

with $(\lambda^*, \mu^*) = (\lambda_1, \mu_1)M_2$. Denote by $G^J_{\mathbf{R}}$ the group of real points of $G^J_{\mathbf{Q}}$. Denote by $\mathcal{D}$ the product of the upper half plane $\mathfrak{H}$ and $\mathbf{C}^l$, the space of complex column vectors of size l: $\mathcal{D} = \mathfrak{H} \times \mathbf{C}^l$. The Jacobi group $G^J_{\mathbf{R}}$ acts on $\mathcal{D}$ in the following manner:

$$g(\tau, z) = \left(M\tau, \frac{z + \lambda\tau + \mu}{J(M, \tau)}\right)$$

$$(g = (M, (\lambda, \mu), \rho) \in G^J_{\mathbf{R}},\ (\tau, z) \in \mathcal{D}),$$

where $J(M, \tau) = c\tau + d$ for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.
Let S be a positive definite half integral symmetric matrix of size l. We define a factor of automorphy $J_{k,S}(g, (\tau, z))$ associated to S and a half integer k by

$$J_{k,S}(g, (\tau, z)) = J(M, \tau)^k e\left(-\mathrm{tr}(S\rho) - \tau S[\lambda] - 2S(\lambda, z) + \frac{c}{J(M,\tau)} S[z + \lambda\tau + \mu]\right)$$

$$\left(g = (M, (\lambda, \mu), \rho) \in G^J_{\mathbf{R}},\ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix},\ (\tau, z) \in \mathcal{D}\right),$$

where the branch of $J(M, \tau)^k = \exp(k \log J(M, \tau))$ is chosen so that $-\pi < \arg J(M, \tau) \leq \pi$. Let Γ be a congruence subgroup of $SL_2(\mathbf{Z})$ having the element -1_2 and Γ^J the subgroup of $G^J_{\mathbf{Q}}$ given by

$$\Gamma^J = \{(M, (\lambda, \mu), \rho) \mid M \in \Gamma, \lambda, \mu \in \mathbf{Z}^l,\ \rho \in Sym_l(\mathbf{Z})\},$$

where $\mathbf{Z}^l$ (resp. $Sym_l(\mathbf{Z})$) denotes the $\mathbf{Z}$-lattice consisting of integral column vectors (resp. integral symmetric matrices) in $\mathbf{Q}^l$ (resp. $Sym_l(\mathbf{Q})$). For any function $\phi : \mathcal{D} \longrightarrow \mathbf{C}$ and $g = (M, (\lambda, \mu), \rho) \in G^J_{\mathbf{R}}$, we set

$$(\phi|_{k,S}g)(\tau, z) = J_{k,S}(g, (\tau, z))^{-1} \phi(g(\tau, z)),$$

$$(\phi|^*_{k,S}g)(\tau, z) = J_{0,S}(g, (\tau, z))^{-1} (\overline{J(M, \tau)})^{-k+l} |J(M, \tau)|^{-l} \phi(g(\tau, z)).$$

In the definition of the latter $(\phi|^*_{k,S}g)$, we may assume that k is an integer, since only such cases can occur in the discussion later on. If k is an integer, then these operations satisfy

$$\phi|_{k,S}g_1g_2 = \phi|_{k,S}g_1|_{k,S}g_2$$

and

$$\phi|^*_{k,S}g_1g_2 = \phi|^*_{k,S}g_1|^*_{k,S}g_2.$$

Note that $\mathfrak{H} \cup \{\infty\} \cup \mathbf{Q}$ is the total set of cusps of Γ. For each element M of Γ, put $M\infty = \zeta$. Denote by Γ_ζ the stabilizer group of ζ in Γ: $\Gamma_\zeta = \{\sigma \in \Gamma \mid \sigma\zeta = \zeta\}$. There exists a unique positive integer N such that the group $M^{-1}\Gamma_\zeta M$ of $SL_2(\mathbf{Z})$ is generated by -1_2 and $\begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix}$. Let k be a positive integer. Now we define the space $J_{k,S}(\Gamma)$ (resp. $J^*_{k,S}(\Gamma)$) of holomorphic (resp. skew-holomorphic) Jacobi forms of index S and weight k with respect to Γ^J. We define $J_{k,S}(\Gamma)$ (resp. $J^*_{k,S}(\Gamma)$) to be the space consisting of all functions $\phi : \mathcal{D} \longrightarrow \mathbf{C}$ which satisfy the following three conditions:

(i) $\phi(\tau, z)$ is holomorphic in τ and z
(resp. $\phi(\tau, z)$ is a smooth function in τ and holomorphic in z)

(ii) $\phi(\tau, z)$ satisfies the identity

$$\phi|_{k,S}\gamma = \phi \qquad (\text{resp. } \ \phi|^*_{k,S}\gamma = \phi) \qquad \text{for } \ \forall\, \gamma \in \Gamma^J$$

(iii) The function $\phi|_{k,S}M$ (resp. $\phi|^*_{k,S}M$) for any $M \in SL_2(\mathbf{Z})$ has a Fourier Jacobi expansion of the form

$$(\phi|_{k,S}M)(\tau, z) = \sum_{\substack{n\in\mathbf{Z}, r\in\mathbf{Z}^l \\ 4n - N{}^t rS^{-1}r \geq 0}} c(n, r) e\left(\frac{n\tau}{N} + {}^t rz\right)$$

$$\left(\text{resp. } \ (\phi|^*_{k,S}M)(\tau, z) = \sum_{\substack{n\in\mathbf{Z}, r\in\mathbf{Z}^l \\ 4n - N{}^t rS^{-1}r \leq 0}} c(n, r) e\left(\frac{n\overline{\tau}}{N} + \frac{i\eta}{2}({}^t rS^{-1}r) + {}^t rz\right)\right),$$

where $\eta = \mathrm{Im}\tau$ and a positive integer N is chosen for each M in the above manner. In the above (iii), M ($\in SL_2(\mathbf{Z})$) is identified with the element $(M, (0, 0), 0)$ in $G^J_{\mathbf{R}}$.

Denote by $J^{cusp}_{k,S}(\Gamma)$ (resp. $J^{*cusp}_{k,S}(\Gamma)$) the subspace of cusp forms of $J_{k,S}(\Gamma)$ (resp. $J^*_{k,S}(\Gamma)$) consisting of all Jacobi forms $\phi \in J_{k,S}(\Gamma)$ (resp. all skew-holomorphic Jacobi forms $\phi \in J^*_{k,S}(\Gamma)$) whose Fourier coefficients $c(n, r)$ in the above (iii) equals zero if $4n - N{}^t rS^{-1}r = 0$.

Let $\Delta \subseteq G^J_{\mathbf{Q}}$ be a finite union of double cosets with respect to Γ^J: $\Delta = \sum_j \Gamma^J \sigma_j \Gamma^J$ $(\sigma_j \in G^J_{\mathbf{Q}})$. Following Skoruppa-Zagier [S-Z2], we define an operator $H_{k,S,\Gamma}(\Delta)$ (resp. $H^{skew}_{k,S,\Gamma}(\Delta)$) acting on $J_{k,S}(\Gamma)$ (resp. $J^*_{k,S}(\Gamma)$) by

$$\phi|H_{k,S,\Gamma}(\Delta) = \sum_{\xi\in\Gamma^J\backslash\Delta} \phi|_{k,S}\xi$$

$$\left(\text{resp. } \ \phi|H^{skew}_{k,S,\Gamma}(\Delta) = \sum_{\xi\in\Gamma^J\backslash\Delta} \phi|^*_{k,S}\phi \right),$$

where the summation is taken over a complete set of representatives ξ for the left Γ^J-cosets of Δ. The operator $H_{k,S,\Gamma}(\Delta)$ (resp. $H^{skew}_{k,S,\Gamma}(\Delta)$) is well-defined and maps $J_{k,S}(\Gamma)$ (resp. $J^*_{k,S}(\Gamma)$) to $J_{k,S}(\Gamma)$ (resp. $J^*_{k,S}(\Gamma)$) and cusp forms to cusp forms (see Proposition 1.1 of [S-Z2]). For L-functions associated with common eigen Jacobi forms in this situation we refer the reader to Sugano [Su].

3 An operator acting on the space of theta series

Let S be a positive definite half-integral symmetric matrix of size l as before and R_S denote the $\mathbf{Z}$-module $(2S)^{-1}\mathbf{Z}^l/\mathbf{Z}^l$. Set

$$d = \det(2S) = \sharp(R_S).$$

We write, for simplicity,

$$S(u,v) = {}^t uSv \quad \text{and} \quad S[u] = {}^t uSu \quad \text{for } u, v \in \mathbf{C}^l.$$

Denote by $V = \mathbf{C}^d$ the $\mathbf{C}$-vector space consisting of column vectors $(x_r)_{r \in R_S}$ $(x_r \in \mathbf{C})$. Let $<x, y>_S$ be the positive definite hermitian scalar product given by

$$<x,y>_S = \sum_{r \in R_S} x_r \overline{y_r} \qquad (x = (x_r)_{r \in R_S}, y = (y_r)_{r \in R_S} \in V).$$

For each $r \in (2S)^{-1}\mathbf{Z}^l$, we define a theta series $\theta_r(\tau, z)$ to be the sum

$$\sum_{q \in \mathbf{Z}^l} e(\tau S[q+r] + 2S(q+r, z)) \quad ((\tau, z) \in \mathcal{D}).$$

Since $\theta_{r+\mu}(\tau, z) = \theta_r(\tau, z)$ for any $\mu \in \mathbf{Z}^l$, one can define $\theta_r(\tau, z)$ for each $r \in R_S$. For each $\tau \in \mathfrak{H}$, let $\Theta_{S,\tau}$ denote the space of holomorphic functions $\theta : \mathbf{C}^l \longrightarrow \mathbf{C}$ with the property

$$\theta(z + \lambda\tau + \mu) = e(-\tau S[\lambda] - 2S(\lambda, z))\theta(z).$$

It is known that $\{\theta_r(\tau, z)\}_{r \in R_S}$ forms a basis of the space $\Theta_{S,\tau}$. For each element $X = (\lambda, \mu) \in \mathbf{Q}^l \times \mathbf{Q}^l$, we denote by $[X]$ the element $(1_2, X, 0)$ of $G^J_{\mathbf{Q}}$. We set

$$\begin{aligned} L &= \mathbf{Z}^l \times \mathbf{Z}^l, \\ H_{\mathbf{Z}} &= \{(1_2, X, \rho) \mid X \in L,\ \rho \in Sym_l(\mathbf{Z})\}. \end{aligned}$$

Then, $H_{\mathbf{Z}}$ is a subgroup of $G^J_{\mathbf{Q}}$. For each $\xi \in G^J_{\mathbf{Q}}$, denote by L_ξ the sublattice $\{X \in L \mid \xi[X]\xi^{-1} \in H_{\mathbf{Z}}\}$ of L. Following Skoruppa-Zagier [S-Z2], we define an operator $U_S(\xi)$ acting on $\Theta_{S,\tau}$ as follows:

$$\theta|U_S(\xi) = \left(\sum_{X \in L_\xi \backslash L} \theta|_{l/2,S}\,\xi[X]\right) \times \frac{1}{[L : L_\xi]}.$$

For this operator Skoruppa-Zagier (Proposition 4.1 of [S-Z2]) proved the following.

Theorem .1 (Skoruppa-Zagier) (i) *For each* $\theta \in \Theta_{S,\tau}$ *and* $\xi \in G^J_{\mathbf{Q}}$, $\theta | U_S(\xi) \in \Theta_{S,\tau}$
(ii) *We arrange* θ_r, $\theta_r | U_S(\xi)$, $(r \in R_S)$ *as column vectors of* $\mathbf{C}^d$. *Then there exists a matrix* $U_S(\xi)$ *of size d (or a linear transformation of* $V = \mathbf{C}^d$*) such that*

$$(\theta_r | U_S(\xi))_{r \in R_S} = U_S(\xi)(\theta_r)_{r \in R_S}, \tag{2.1}$$

where $U_S(\xi)$ *is independent of the choice of* $\tau \in \mathfrak{H}$.

Remark. (1) For the matrix $U_S(\xi)$, we have used the same notation as for the operator $U_S(\xi)$ by abuse of notation.
(2) If $\xi = (M, 0, 0)$ and $M \in SL_2(\mathbf{Z})$, then the identity (2.1) is nothing but *the theta transformation formula*:

$$(\theta_r(M(\tau, z)))_{r \in R_S} = J_{l/2,S}(M, (\tau, z))U_S(M)(\theta_r(\tau, z))_{r \in R_S} \quad (\forall M \in SL_2(\mathbf{Z})),$$

where $M(\tau, z) = (M\tau, \frac{z}{c\tau+d})$ and $U_S(M) = U_S((M, 0, 0))$ in this case is a unitary matrix with respect to the inner product $< \ , \ >_S$.

4 Where does $U_S(\xi)$ come from?

Let k be a positive integer and put $\kappa = (k - l/2)/2$. We define a factor of automorphy $j_M(\tau)$ by

$$j_M(\tau) = \exp(2i\kappa \arg J(M, \tau)).$$

Denote by $\mathcal{M}_{S,k-l/2}(\Gamma)$ the space of all functions $f : \mathfrak{H} \longrightarrow V$ satisfying the following conditions

(i) $\eta^{-\kappa} f(\tau)$ is holomorphic on $\mathfrak{H}$ and also finite at any cusps of Γ

(ii) $f(M\tau) = \overline{U_S(M)} j_M(\tau) f(\tau)$ for any $M \in \Gamma$.

Since each Jacobi form $\phi(\tau, z)$ of $J_{k,S}(\Gamma)$ is an element of $\Theta_{S,\tau}$ as a function of z, $\phi(\tau, z)$ has an expression as a linear combination of θ_r's:

$$\phi(\tau, z) = \sum_{r \in R_S} \eta^{-\kappa} f_r(\tau) \theta_r(\tau, z).$$

Then the collection $f(\tau) = (f_r(\tau))_{r \in R_S}$ is a modular form of $\mathcal{M}_{S,k-l/2}(\Gamma)$. It is well-known that $J_{k,S}(\Gamma)$ is isomorphic to $\mathcal{M}_{S,k-l/2}(\Gamma)$ as $\mathbf{C}$-linear spaces via the correspondence $\iota : \phi \longrightarrow f = (f_r)_{r \in R_S}$. Let $\Delta \subseteq G^J_{\mathbf{Q}}$ be a finite union of Γ^J-double cosets. Let $p : G^J_{\mathbf{Q}} \longrightarrow SL_2(\mathbf{Q})$ denote the natural projection map. For each A of $p(\Delta)$ we put

$$V_\Delta(A) = \sum_{\xi \in H_{\mathbf{Z}} \backslash p^{-1}(A) \cap \Delta / H_{\mathbf{Z}}} [L : L_\xi] U_S(\xi),$$

where the summation is over a complete set of representatives ξ of the double cosets of $p^{-1}(A)\cap\Delta$ with respect to $H_{\mathbf{Z}}$ (this is a finite sum). Then this quantity $V_\Delta(A)$ is well-defined. If $\Delta=\Gamma^J$, then $V_\Delta(A)$ equals the linear operator $U_S(A)=U_S((A,0.0))$. The action of Hecke operators $H_{k,S,\Gamma}(\Delta)$ on $J_{k,S}(\Gamma)$ is transferred in terms of modular forms of $\mathcal{M}_{S,k-l/2}(\Gamma)$. There exists a linear operator $\widetilde{H}_{k,S,\Gamma}(\Delta)$ acting on $\mathcal{M}_{S,k-l/2}(\Gamma)$ such that $\iota\circ H_{k,S,\Gamma}(\Delta)=\widetilde{H}_{k,S,\Gamma}(\Delta)\circ\iota$. Then we easily have

$$(f|\widetilde{H}_{k,S,\Gamma}(\Delta))(\tau)=\sum_{A\in\Gamma\backslash p(\Delta)}{}^tV_\Delta(A)j_A(\tau)^{-1}f(A\tau)\qquad(f\in\mathcal{M}_{S,k-l/2}(\Gamma)),$$

where A runs over a complete set of representatives of the left Γ-cosets of $p(\Delta)$ and the sum is well-defined.

In this manner the operator $U_S(\xi)$ is coming in our sight. It seems that $U_S(\xi)$ is a very attractive arithmetic object.

5 Selberg type zeta functions

For $M\in SL_2(\mathbf{Z})$, we write $U_S(M)$ instead of $U_S((M,0,0))$ in (2.1). We set

$$R_S^0=\{r\in R_S\mid r\equiv -r\pmod{\mathbf{Z}^l}\}.$$

Since $U_S(-1_2)$ has eigen values $\pm e^{-\pi il/2}$ (see (1.6) of [Ar]), it has the block decomposition

$$U_S(-1_2)=e^{-\pi il/2}Q\begin{pmatrix}1_{d(+)}&0\\0&-1_{d(-)}\end{pmatrix}Q^{-1},\tag{4.1}$$

where Q is a certain unitary matrix of size d and $d(+)=(d+d_0)/2$ (resp. $d(-)=(d-d_0)/2$). We easily have

$$V_\Delta(A)U_S(-1_2)=U_S(-1_2)V_\Delta(A)\quad\text{for any }A\in p(\Delta).$$

Therefore, $V_\Delta(A)$ has the block decomposition similar to (4.1):

$$V_\Delta(A)=Q\begin{pmatrix}V_\Delta^+(A)&0\\0&V_\Delta^-(A)\end{pmatrix}Q^{-1}\tag{4.2}$$

with $V_\Delta^+(A)$ (resp. $V_\Delta^-(A)$) a matrix of size $d(+)$ (resp. $d(-)$).For $A\in SL_2(\mathbf{Q})$, let $Z_\Gamma(A)$ denote the centralizer of A in Γ. Denote by $Hyp^+(\Delta)$ the set of hyperbolic elements P of $p(\Delta)$ with $\mathrm{tr}P>2$ which do not fix any cusps of Γ. We set, for $\varepsilon=\pm$,

$$\zeta_{\Delta,S,\varepsilon}(s)=\sum_{P\in Hyp^+(\Delta)/\!\!/\Gamma}\mathrm{tr}V_\Delta^\varepsilon(P)\log N(P_0)\times\frac{N(P)^{-s}}{1-N(P)^{-1}},$$

where $Hyp^+(\Delta)/\!/\Gamma$ denote a complete set of representatives of the Γ-conjugacy classes of elements of $Hyp^+(\Delta)$, and where, for each $P \in Hyp^+(\Delta)$, P_0 together with the element -1_2 is the generator of the centralizer $Z_\Gamma(P)$. It can be shown that $\zeta_{\Delta,S,\varepsilon}(s)$ is absolutely convergent for $\mathrm{Re}(s) > 1$. If $\Delta = \Gamma^J$, then, $\zeta_{\Delta,S,\varepsilon}(s)$ coincides with the logarithmic derivative of the Selberg zeta function associated with Γ, S:

$$\zeta_{\Delta,S,\varepsilon}(s) = (Z'_{\Gamma,S,\varepsilon}/Z_{\Gamma,S,\varepsilon})(s),$$

where $\varepsilon = \pm$ and

$$Z_{\Gamma,S,\varepsilon}(s) = \prod_{\{P_0\}_\Gamma, \mathrm{tr}P_0>2} \prod_{n=0}^{\infty} \det(1_{d(\varepsilon)} - U_S^\varepsilon(P_0)N(P_0)^{-s-n}),$$

P_0 running over the Γ-conjugacy classes of primitive hyperbolic elements of Γ with $\mathrm{tr}P_0 > 2$. Here, $U_S^\pm(A)$ ($A \in SL_2(\mathbf{Z})$) is defined similarly as in (4.2) from $U_S(A)$. For details concerning the Selberg zeta functions $Z_{\Gamma,S,\varepsilon}(s)$ we refer to [Ar]. Via the theory of general Selberg trace formula the Selberg type zeta functions $\zeta_{\Delta,S,\varepsilon}(s)$ can be analytically continued to a meromorphic function of s in the whole complex plane. This analytic continuation is crucial to the calculation of the traces of Hecke operators.

6 Traces of Hecke operators

Let Δ be as before. Each elliptic element R of $SL_2(\mathbf{R})$ is $SL_2(\mathbf{R})$-conjugate to some $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ with $0 < \theta < 2\pi$, where θ is uniquely determined by R. We often write $\theta(R)$ for this θ. Denote by $Ell^+(\Delta)$ the set of all elliptic elements R of $p(\Delta)$ with $0 < \theta(R) < \pi$. Denote by $Ell^+(\Delta)/\!/\Gamma$ a complete set of representatives of the Γ-conjugacy classes of all elements of $Ell^+(\Delta)$. Let $\zeta_1, \zeta_2, \dots, \zeta_h$ be a complete set of representatives of the Γ-equivalence classes of cusps of Γ. For each j $(1 \le j \le h)$, one can choose an element $A_j \in SL_2(\mathbf{R})$ such that -1_2 and $T_j := A_j^{-1}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}A_j$ generate the stabilizer group Γ_{ζ_j} of the cusp ζ_j. For each j $(1 \le j \le h)$, denote by $Hyp_j^+(\Delta)$ the set of all hyperbolic elements P of $p(\Delta)$ with $\mathrm{tr}P > 2$ and $P\zeta_j = \zeta_j$. The set $Hyp_j^+(\Delta)$ is atable under the conjugation by any element of Γ_{ζ_j}. Denote by $Hyp_j^+(\Delta)/\!/\Gamma_{\zeta_j}$ a complete set of representatives of the Γ_{ζ_j}-conjugacy classes of all elements of $Hyp_j^+(\Delta)$. Moreover for each j $(1 \le j \le h)$, we denote by $Par_j^+(\Delta)$ the set of all parabolic elements P of $p(\Delta)$ satisfying the conditions $\mathrm{tr}P = 2$, $P\zeta_j = \zeta_j$ and $P \neq 1_2$. Let $N = 4\det(2S) = 4d$ and $\Gamma(N)$ the principal congruence subgroup of $SL_2(\mathbf{Z})$ with level N. Set, for each j $(1 \le j \le h)$,

$$\Gamma_j^+ = \Gamma_{\zeta_j} \cap \Gamma(N).$$

Then the group Γ_j^+ is generated by $T_j^{n_j}$ with a positive integer n_j. This integer n_j is uniquely determined. We call two elements A, B of $Par_j^+(\Delta)$ Γ_j^+-equivalent, if there exists an element M of Γ_j^+ with $B = MA$. Denote by $Par_j^+(\Delta)/\Gamma_j^+$ a complete set of representatives of the Γ_j^+-equivalence classes of all elements of $Par_j^+(\Delta)$. Each element P of $Par_j^+(\Delta)$ has an expression

$$P = A_j^{-1}\begin{pmatrix} 1 & r(P) \\ 0 & 1 \end{pmatrix} A_j$$

with a uniquely determined rational number $r(P) \in \mathbb{Q}$, $r(P) \neq 0$. If P is a representative of $Par_j^+(\Delta)/\Gamma_j^+$, $r(P)$ is uniquely determined modulo n_j.
Let k be an integer and set

$$\kappa = (k - l/2)/2.$$

Denote by $\varepsilon(k)$ the sign $+$ or $-$ according as k is even or not. We set

$$\begin{aligned}
C_{\Gamma,\Delta}(k) &= \frac{1}{4\pi} vol(\Gamma\backslash\mathfrak{H})\mathrm{tr}(V_\Delta^{\varepsilon(k)}(1_2))(2\kappa - 1) \\
&+ \sum_{R \in Ell^+(\Delta)/\!\!/\Gamma} \mathrm{tr}(V_\Delta^{\varepsilon(k)}(R)) \times \frac{e^{-2i\kappa\theta(R) + i\theta(R)}}{\nu(e^{i\theta(R)} - e^{-i\theta(R)})} \\
&- \frac{1}{2}\sum_{j=1}^{h} \sum_{P \in Hyp_j^+(\Delta)/\!\!/\Gamma_{\zeta_j}} \mathrm{tr}(V_\Delta^{\varepsilon(k)}(P)) \times \frac{N(P)^{-\kappa}}{1 - N(P)^{-1}} \\
&- \sum_{j=1}^{h} \sum_{P \in Par_j^+(\Delta)/\Gamma_j^+} \frac{1}{2n_j}\mathrm{tr}(V_\Delta^{\varepsilon(k)}(P)) \times \begin{cases} 1 - i\cot\frac{\pi r(P)}{n_j} & \cdots r(P) \not\equiv 0 \bmod n_j \\ 1 & \cdots r(P) \equiv 0 \bmod n_j \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
C_{\Gamma,\Delta}^*(k) &= \frac{1}{4\pi} vol(\Gamma\backslash\mathfrak{H})\mathrm{tr}(V_\Delta^{\varepsilon(k)}(1_2))(-2\kappa - 1) \\
&+ \sum_{R \in Ell^+(\Delta)/\!\!/\Gamma} \mathrm{tr}(V_\Delta^{\varepsilon(k)}(R)) \times \frac{e^{-2i\kappa\theta(R) - i\theta(R)}}{\nu(e^{i\theta(R)} - e^{-i\theta(R)})} \\
&- \frac{1}{2}\sum_{j=1}^{h} \sum_{P \in Hyp_j^+(\Delta)/\!\!/\Gamma_{\zeta_j}} \mathrm{tr}(V_\Delta^{\varepsilon(k)}(P)) \times \frac{N(P)^{\kappa}}{1 - N(P)^{-1}} \\
&- \sum_{j=1}^{h} \sum_{P \in Par_j^+(\Delta)/\Gamma_j^+} \frac{1}{2n_j}\mathrm{tr}(V_\Delta^{\varepsilon(k)}(P)) \times \begin{cases} 1 + i\cot\frac{\pi r(P)}{n_j} & \cdots r(P) \not\equiv 0 \bmod n_j \\ 1 & \cdots r(P) \equiv 0 \bmod n_j, \end{cases}
\end{aligned}$$

where

$$vol(\Gamma\backslash\mathfrak{H}) = \int_{\Gamma\backslash\mathfrak{H}} \eta^{-2} d\xi d\eta \quad (\xi = \mathrm{Re}\tau,\ \eta = \mathrm{Im}\tau).$$

We denote by $\mathrm{tr}(H_{k,S,\Gamma}(\Delta), J_{k,S}(\Gamma))$ the trace of the action of $H_{k,S,\Gamma}(\Delta)$ on $J_{k,S}(\Gamma)$ and so on. Let $\Theta_{S,\Gamma}$ denote the space of theta functions $\theta(\tau, z)$ satisfying the following conditions:

(i) $\theta(\tau, z)$, as a function of z, is an element of $\Theta_{S,\tau}$

(ii) $\theta|_{l/2,S}M = \theta$ for any $M \in \Gamma$.

Then, $\Theta_{S,\Gamma}$ is isomorphic to the space $J_{l/2,S}(\Gamma)$ of Jacobi forms of weight $l/2$ with respect to Γ^J. The Hecke operator $H_{l/2,S,\Gamma}(\Delta)$ operates on $\Theta_{S,\Gamma}$. We have the following theorem.

Theorem .2 *Assume that Γ is a congruence subgroup of $SL_2(\mathbf{Z})$ having the element -1_2. Let k be an integer and $\Delta \subseteq G^J_{\mathbf{Q}}$ a finite union of Γ^J-double cosets.*

(i) *If $k > l/2 + 2$, then,*

$$\mathrm{tr}(H_{k,S,\Gamma}(\Delta), J^{cusp}_{k,S}(\Gamma)) = C_{\Gamma,\Delta}(k).$$

If $k < l/2 - 2$, then,

$$\mathrm{tr}(H^{skew}_{l-k,S,\Gamma}(\Delta), J^{*cusp}_{l-k,S}(\Gamma)) = C^*_{\Gamma,\Delta}(k).$$

(ii) *Assume that l is odd. Denote by ε the sign $+$ or $-$ according as l is congruent to* 1 *or* 3 *modulo* 4 (i.e., $\varepsilon = \varepsilon((l-1)/2)$.

If $k = (l+3)/2$, then,

$$\mathrm{tr}(H_{k,S,\Gamma}(\Delta), J^{cusp}_{k,S}(\Gamma)) = \mathrm{Res}_{s=3/4}\zeta_{\Delta,S,\varepsilon}(s) + C_{\Gamma,\Delta}(k).$$

If $k = (l+1)/2$, then,

$$\mathrm{tr}(H_{k,S,\Gamma}(\Delta), J_{k,S}(\Gamma)) = \mathrm{Res}_{s=3/4}\zeta_{\Delta,S,-\varepsilon}(s).$$

If $k = (l-1)/2$, then,

$$\mathrm{tr}(H^{skew}_{l-k,S,\Gamma}(\Delta), J^*_{l-k,S}(\Gamma)) = \mathrm{Res}_{s=3/4}\zeta_{\Delta,S,\varepsilon}(s).$$

If $k = (l-3)/2$, then,

$$\mathrm{tr}(H^{skew}_{l-k,S,\Gamma}(\Delta), J^{*cusp}_{l-k,S}(\Gamma)) = \mathrm{Res}_{s=3/4}\zeta_{\Delta,S,-\varepsilon}(s) + C^*_{\Gamma,\Delta}(k).$$

(iii) *Assume that l is even. Let $\varepsilon = \varepsilon(l/2)$.*

If $k = l/2 + 2$, then,

$$\mathrm{tr}(H_{k,S,\Gamma}(\Delta), J^{cusp}_{k,S}(\Gamma)) = \mathrm{Res}_{s=1}\zeta_{\Delta,S,\varepsilon}(s) + C_{\Gamma,\Delta}(k).$$

If $k = l/2 - 2$, then,

$$\mathrm{tr}(H^{skew}_{l-k,S,\Gamma}(\Delta), J^{*cusp}_{l-k,S}(\Gamma)) = \mathrm{Res}_{s=1}\zeta_{\Delta,S,\varepsilon}(s) + C^*_{\Gamma,\Delta}(k).$$

If $k = l/2$, then,

$$\mathrm{tr}(H_{l/2,S,\Gamma}(\Delta), \Theta_{S,\Gamma}) = \mathrm{Res}_{s=1}\zeta_{\Delta,S,\varepsilon}(s).$$

For the proof we use Fischer's resolvent trace formula [Fi] and the method of Skoruppa-Zagier [S-Z2]. We can deduce the following corollary from (ii), (iii) of the above theorem.

Corollary .3 (i) *Assume that l is odd. Then,*

$$\mathrm{tr}(H_{(l+3)/2,S,\Gamma}(\Delta), J^{cusp}_{(l+3)/2,S}(\Gamma)) = \mathrm{tr}(H^{skew}_{(l+1)/2,S,\Gamma}(\Delta), J^{*}_{(l+1)/2,S}(\Gamma)) + C_{\Gamma,\Delta}\left(\frac{l+3}{2}\right),$$

$$\mathrm{tr}(H^{skew}_{(l+3)/2,S,\Gamma}(\Delta), J^{*cusp}_{(l+3)/2,S}(\Gamma)) = \mathrm{tr}(H_{(l+1)/2,S,\Gamma}(\Delta), J_{(l+1)/2,S}(\Gamma)) + C^{*}_{\Gamma,\Delta}\left(\frac{l-3}{2}\right).$$

(ii) *Assume that l is even. Then,*

$$\mathrm{tr}(H_{l/2+2,S,\Gamma}(\Delta), J^{cusp}_{l/2+2,S}(\Gamma)) = \mathrm{tr}(H_{l/2,S,\Gamma}(\Delta), \Theta_{S,\Gamma}) + C_{\Gamma,\Delta}\left(\frac{l}{2}+2\right),$$

$$\mathrm{tr}(H^{skew}_{l/2+2,S,\Gamma}(\Delta), J^{*cusp}_{l/2+2,S}(\Gamma)) = \mathrm{tr}(H_{l/2,S,\Gamma}(\Delta), \Theta_{S,\Gamma}) + C^{*}_{\Gamma,\Delta}\left(\frac{l}{2}-2\right).$$

Remark. In the case of $l = 1$ the first identity of the above (i) has been already obtained by Skoruppa-Zagier[S-Z1]. The results in Theorem 2 and Corollary 3 are consistent with those of [S-Z1, 2].

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